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LETTER TO THE EDITOR

Critical dynamics below the percolation threshold

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Abstract. The critical dynamics of the diluted two-dimensional Ising model below the percolation threshold is studied. The dynamics exhibits a crossover from a concentration-dependent behaviour for $\xi_T \gg \xi_p$ to a percolating lattice type behaviour for $\xi_T \ll \xi_p$, where ξ_T and ξ_p are the thermal and percolation correlation lengths, respectively. As a consequence, the dynamical exponents A and B are shown to be the limits of dynamical *functions* $A(x)$ and $B(x)$, respectively; here, $x = \ln \xi_p / \ln \xi_T$. Extensive Monte Carlo simulations support our predictions.

The critical dynamics of disordered spin systems have attracted considerable interest in recent years. It is now fairly well established [1–9] that conventional dynamic scaling [10] breaks down for diluted spin systems at the percolation threshold. According to conventional dynamic scaling, the average time, $\tau_{AV}(T)$, scales as

$$\ln \tau_{AV}(T) \sim Z \ln \xi_T \tag{1}$$

where ξ_T is the thermal correlation length and Z is a dynamic critical exponent. At $p = p_c$, the percolation threshold, it is found the Z is in fact a temperature-dependent function

$$Z(T) = A(\ln \xi_T) + B \tag{2}$$

where A and B are two new dynamical exponents. Both analytic [2–4, 7, 8] and computational [5, 6, 9] results confirm the new dynamics for Ising and Potts models at the percolation threshold; estimates of A and B for various two-dimensional models may be found in Jain [9]. Very recently, critical crossover phenomena has been reported by Heuer [11] who investigated site-disordered Ising systems over a large range of dilution above the percolation threshold ($0.6 \leqslant p \leqslant 1.0$).

In this letter we study the critical behaviour of systems *below* the percolation threshold ($p < p_c$). We present both analytical and computational results. By applying the arguments of Harris and Stinchcombe [2], we shall show that for $p < p_c$ we have to introduce dynamical exponents which take the form of scaling functions, A and B being retrieved in appropriate limits. Furthermore, we also report the results of extensive Monte Carlo simulations of the two-dimensional Ising model on a square lattice with bond-dilution which support our analytical predictions. A preliminary account of this work was presented at the Vth Rencontres de Blois on 'Chaos and Complexity' [12].

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The Hamiltonian is given by

$$H=-\sum_{\langle ij\rangle} J_{ij}S_iS_j \tag{3}$$

where Ising spins $S_i=\pm 1$ are situated on every lattice site, the sum runs over nearest neighbours only and the quenched ferromagnetic couplings are selected according to

$$P(J_{ij})=(1-p)\delta(J_{ij})+p\delta(J_{ij}-1) \tag{4}$$

p being the bond concentration. Boltzmann's constant and the nearest-neighbour interaction are both set to unity. The pure system has a second-order phase transition at $T_c(p=1)=2/\ln(1+2^{1/2})$ and, of course, $T_c(p=p_c=1/2)=0$.

The correlation length, $\xi(T,p)$, is given by [11]

$$\xi^{-1}(T,p)=\xi_T^{-1}(T,p_c)+\xi_p^{-1}(0,p) \tag{5}$$

where ξ_T and ξ_p are the thermal and percolation correlation lengths, respectively. At $p=p_c$ the correlation length is given by just the thermal correlation length because the percolation correlation length is infinite. Below p_c, however, we have to take *both* correlation lengths into consideration; further, we have [13]

$$\xi_T=\xi_0\,\mathrm{e}^{2\nu_T/T} \tag{6}$$

where [6] $\xi_0=0.22\pm 0.01$ and

$$\xi_p\sim(p_c-p)^{-\nu_p}. \tag{7}$$

The cross-over exponent [14] is predicted to be 1 and, as a consequence, we can identify [6, 15] $\nu_T=\nu_p 4/3$.

We consider Metropolis dynamics [16] in which the spins are updated according to

$$W(S_i\rightarrow -S_i)=\min\{1,\exp(-\Delta E/T)\} \tag{8}$$

where ΔE is the energy change involved.

We adopt the real-space renormalization group technique of Harris and Stinchcombe [2]. In this approach one considers a honeycomb lattice with configurational disorder and replaces it with a similar lattice dilated by a scale factor $b=2$. ([2] should be consulted for further technical details.) After n iterations, the average relaxation time for a domain wall to traverse a renormalized bond is given by [2]

$$T_{\mathrm{AV}}\sim\left[\frac{\xi_T^n}{b^{n(n-1)/2}}\right]^{1/\nu_T}. \tag{9}$$

If $\xi_T\ll\xi_p$, the iterations stop at $b^n\sim\xi_T$ and we obtain

$$\ln\tau_{\mathrm{AV}}=A(\ln\xi_T)^2+B(\ln\xi_T) \tag{10}$$

with $A=1/2\nu_T\ln b$, and $B=1/2\nu_T$ (+corrections). Equation (10) is, of course, identical to the form obtained at p_c (see equations (1) and (2)).

If $\xi_T\gg\xi_p$, the iterations stop at $b^n\sim\xi_p$ and we get instead of equation (10) that

$$\ln\tau_{\mathrm{AV}}=\frac{1}{2\nu_T\ln b}[2\ln\xi_p\ln\xi_T-(\ln\xi_p)^2+\ln\xi_p\ln b]. \tag{11}$$

Both limits, $\xi_T \ll \xi_p$ and $\xi_T \gg \xi_p$, can be obtained from

$$\ln \tau_{AV} = A(x)(\ln \xi_T)^2 + B(x)(\ln \xi_T) \tag{12}$$

where $x = \ln \xi_p / \ln \xi_T$ and the functional dependence of A and B has been indicated explicitly. The two scaling functions, $A(x)$ and $B(x)$, are such that

$$A(x) = \begin{cases} A & \text{for } x \to \infty \\ \dfrac{2x - x^2}{2\nu_T \ln b} & \text{for } x \ll 1 \end{cases} \tag{13}$$

and

$$B(x) = \begin{cases} B & \text{for } x \to \infty \\ \dfrac{x}{2\nu_T} & \text{for } x \ll 1. \end{cases} \tag{14}$$

We now discuss the results of Monte Carlo simulations of the bond-diluted two-dimensional Ising model on a square 64×64 lattice.

For any given p and temperature, T, the nearest-neighbour interactions are chosen according to equation (4). Periodic boundary conditions are imposed throughout this work. The spins are all pointing up at the start and we perform a conventional Monte Carlo simulation [17] of the system using the transition probability given in equation (8). The data presented here were collected over the ranges $0.25 \leqslant T \leqslant 1.2$ and $0.1 \leqslant p \leqslant 0.3$, and have been averaged over many samples (50–10 000).

The key quantity studied in the simulations is τ_{AV}, the average relaxation time, which is given by

$$\tau_{AV} = \int_0^\infty C(t)\, \mathrm{d}t \tag{15}$$

and here $C(t) = N^{-1} \Sigma_i S_i(t_0) S_i(t + t_0)$, where t_0 is an initial time, $t > t_0$ and, in this particular situation, $N = 4096$. We choose t_0 so that $t_0 \geqslant \tau_0$, the equilibration time of the system.

In figure 1 we show a log–log plot of τ_{AV} against ξ_T for various values of bond concentrations over the range $(0.7 \leqslant \ln \xi_T \leqslant 9.2)$; note that the thermal correlation length has been determined from equation (6). Throughout our simulations we have $\min(\xi_T, \xi_p) \ll$ the linear size of the lattice. Consequently, our results are not expected to be influenced by finite-size effects.

Clearly, for each value of p, the divergence of $\ln \tau_{AV}$ falls into two regimes: an initial nonlinear divergence and an apparently linear divergence, providing ξ_T is sufficiently large. To see if the initial divergence is indeed quadratic as predicted by equation (10), we show a magnified plot of the data for $0.7 \leqslant \ln \xi_T \leqslant 2.3$ in figure 2. (For clarity, we have omitted the data for $p = 0.3$). The best overall quadratic fit in each case is also displayed. It should be remembered that whereas equation (10) is only valid for $\xi_T \ll \xi_p$, the data shown in figure 2 also include those for $\xi_T \sim \xi_p$. As a consequence, one should not be too surprised by the deviations from the quadratic fits, especially for moderately large bond concentrations. Further, it is clear from equation (12) that we expect the quadratics in question to have coefficients $A(x)$ and $B(x)$ which vary with the ratio $\ln \xi_p / \ln \xi_T (= x)$. In figure 3 we plot the coefficients of the quadratic

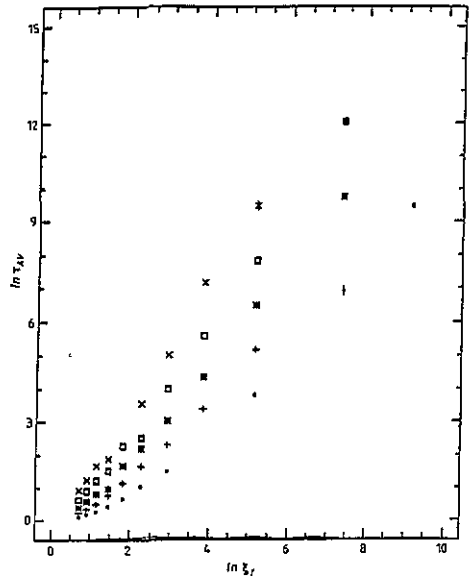

Figure 1. A plot of ln τ_{AV} against ln ξ_T for various bond concentrations p: •, $p=0.1$; +, $p=0.15$; *, $p=0.2$; □, $p=0.25$; ×, $p=0.3$.

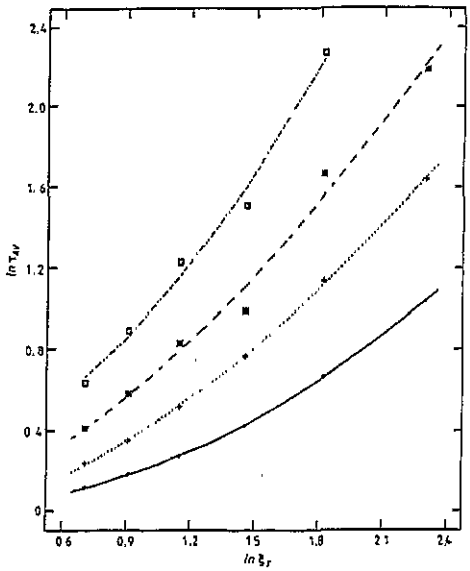

Figure 2. A magnified plot of ln τ_{AV} against ln ξ_T for $0.7 \leqslant \ln \xi_T \leqslant 2.3$. The best quadratic fit in each case is also shown. •, $p=0.1$; +, $p=0.15$; *, $p=0.2$; □, $p=0.25$.

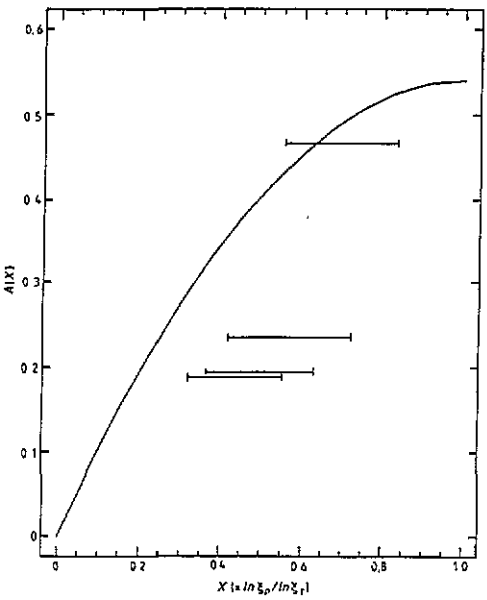

Figure 3. A plot of $A(x)$ against x. Also shown are the values of the coefficients of the quadratic terms for the fits shown in figure 2. (See text.)

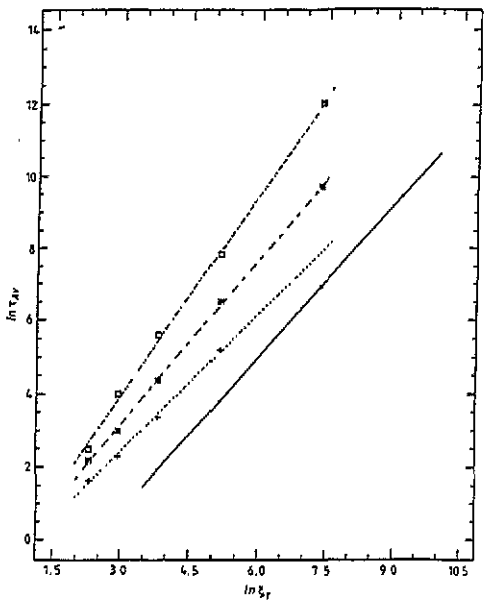

Figure 4. A magnified plot of ln τ_{AV} against ln ξ_T for $2.4 \leqslant \ln \xi_T \leqslant 9.2$. The best linear fit in each case is also shown. •, $p=0.1$; +, $p=0.15$; *, $p=0.2$; □, $p=0.25$.

Table 1. Values of the gradients of the linear fits shown in figure 4 and the corresponding term in equation (11). (See text.)

P	0.1	0.15	0.2	0.25
Gradient	1.41	1.24	1.50	1.84
$\ln \xi_p / \nu_T \ln b$	1.32	1.51	1.74	2.00

Table 2. Values of $x(p)$ at the crossover as determined from figure 1.

p	0.1	0.15	0.2	0.25	0.3
$x^c(p)$	0.24	0.30	0.31	0.40	0.50

terms against x for the fits shown in figure 2, the range of x values in each case being determined by the range over which each fit is valid. Also shown in figure 3 is the predicted form of $A(x)$ as determined by equation (13). Clearly, our numerical values are in good qualitative agreement with the theory. The quantitative discrepancy can be understood by the reservations outlined above.

In the other regime, namely $\xi_T \gg \xi_p$, we expect equation (11) to hold. In figure 4 we show a magnified plot of the data for $2.4 \leqslant \ln \xi_T \leqslant 9.2$. Here the divergence of $\ln \tau_{AV}$ is evidently linear as predicted. The line of best fit in each case is also shown and the gradients are given in table 1. These should be compared with the theoretical values which are also given in table 1.

The crossover between the two regimes occurs at $x^c(p)$ (where $x = \ln \xi_p / \ln \xi_T$), which, as determined from figure 1, would appear to be a monotonically increasing function of p (see table 2).

To conclude, we have studied the critical dynamics of the diluted two-dimensional Ising model below the percolation threshold. We have shown that there is a crossover at $x^c(p)$ from a concentration-dependent behaviour for $\xi_T \gg \xi_p$ to a percolating lattice type behaviour for $\xi_T \ll \xi_p$; $x^c(p)$ would appear to be a monotonically increasing function of p. Furthermore, the dynamical exponents A and B are actually the limits of dynamical functions $A(x)$ and $B(x)$, respectively. Our numerical results are consistent with the theory.

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